4D-Variational Data Assimilation for Locally Nested Numerical Ocean Models

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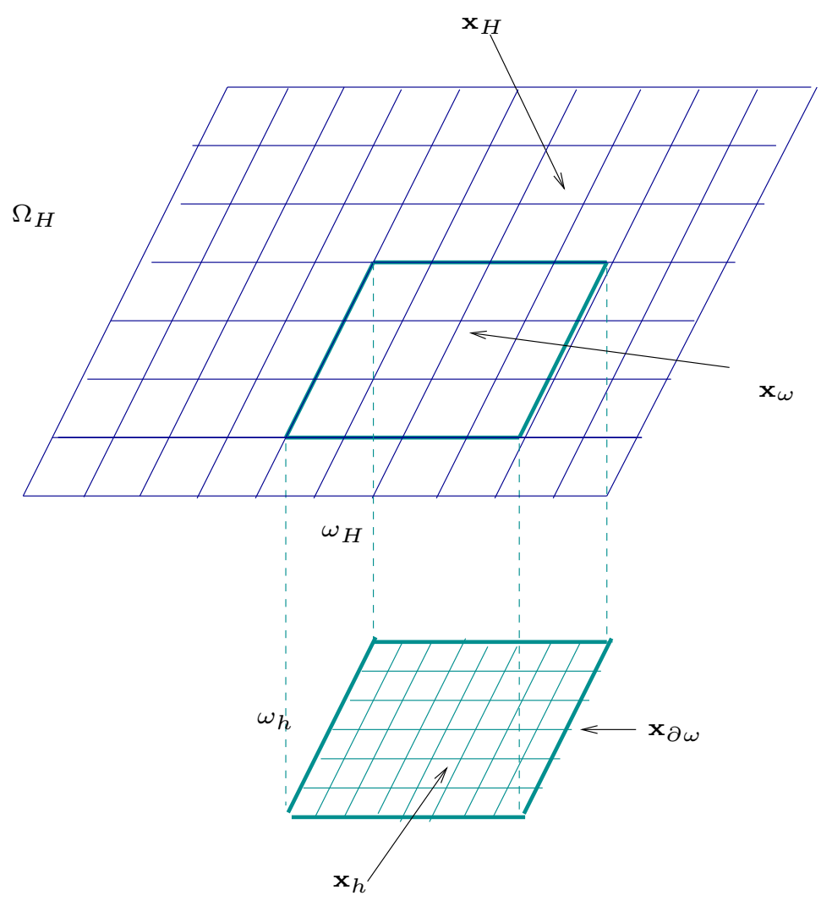
ABSTRACT

A common way to develop **regional ocean modeling systems** consists in **embedding a high resolution local model into a coarse resolution model** covering a larger domain. The local model then takes its boundary conditions from its parent model (one-way interaction), while the parent model solution may additionally be periodically updated using the local fine resolution solution (two-way interaction).
However, for data assimilation purposes, the multi-resolution structure of the modeling system is generally ignored. One assimilates data either in one of the two models only, or in both models separately, but without properly taking into account the interactions between the two numerical solutions.
In this poster, we address the problem of **4D variational data assimilation** in such locally nested models, for the control of the initial conditions on both models. The adjoint system is derived in both cases of one-way and two-way interactions. It is shown that the adjoint formulation adds new interactions between the grids, in the opposite direction of the interactions existing in the direct formulation. In particular, in the one-way case, the adjoint formulation creates a retroaction term from the fine grid onto the coarse grid. The design of the multi-grid background error covariance matrix is also discussed, as well as the addition of a new control variable corresponding to the errors in the interactions between the coarse resolution and fine resolution solutions.
These formulations are illustrated and discussed in the idealized test case of a 2D shallow water model. In particular, it is shown that this multi-resolution approach leads to improved results with regard to the usual method consisting in the assimilation of data on the local fine resolution model only, with a control of its corresponding initial condition and of its boundary values

1. Formulation of the nested models

We consider the general case of a high resolution model, covering the local domain ω, embedded in a coarser resolution model covering the larger domain Ω. The local high resolution grid and the global coarse resolution grid are denoted respectively ω_h and Ω_H. The corresponding state vectors are denoted respectively $\mathbf{x}_h$ and $\mathbf{x}_H$. We also denote ω_H the part of the grid Ω_H corresponding to the local domain ω.



One-way Interaction

The coarse grid model provides **boundary conditions** to the high resolution model.

Domain Ω_H

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{x}_H}{\partial t} = F(\mathbf{x}_H) \quad \text{on} \quad \Omega_H \times [0,T] \\ \mathbf{x}_H(t=0) = \mathbf{x}_H^0 \end{array}\right.$$

Domain ω_h

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{x}_h}{\partial t} = F(\mathbf{x}_h, \mathbf{x}_{\partial\omega}) \quad \text{on} \quad \omega_h \times [0,T] \\ \mathbf{x}_h(t=0) = \mathbf{x}_h^0 \\ \mathbf{x}_{\partial\omega} = I_H^h(\mathbf{x}_H) \quad \text{on} \quad \partial\omega_h \times [0,T] \end{array}\right.$$

Two-way Interaction

A **feedback term** from the fine grid onto the coarse grid is added.

Domain Ω_H

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{x}_H}{\partial t} = F(\mathbf{x}_H, \mathbf{x}_\omega) \quad \text{on} \quad \Omega_H \times [0,T] \\ \mathbf{x}_H(t=0) = \mathbf{x}_H^0 \\ \mathbf{x}_\omega = G_h^H(\mathbf{x}_h) \quad \text{on} \quad \mathring{\omega}_H \times [0,T] \end{array}\right.$$

Domain ω_h

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{x}_h}{\partial t} = F(\mathbf{x}_h, \mathbf{x}_{\partial\omega}) \quad \text{on} \quad \omega_h \times [0,T] \\ \mathbf{x}_h(t=0) = \mathbf{x}_h^0 \\ \mathbf{x}_{\partial\omega} = I_H^h(\mathbf{x}_H) \quad \text{on} \quad \partial\omega_h \times [0,T] \end{array}\right.$$

2. Variational Data Assimilation

Observations and observation operator

The vectors of observations at time t_i that will be assimilated on Ω_H and ω_h are denoted respectively $\mathbf{y}_H^i$ and $\mathbf{y}_h^i$. Discrete observation operators and observation error covariance matrices are defined on the coarse grid (H_H^i and $\mathbf{R}_H^i$) and on the fine grid (H_h^i and $\mathbf{R}_h^i$) in the same way as in the context of a single grid. Note that an observation located in ω can possibly be assimilated on both grids. This question of the optimal distribution of the observations on the grids is important in the context of multiresolution systems. However it is out of the scope of the present poster, and is not adressed in the following.

State vector

The state vector $\mathbf{x}$ of the two-grid model is composed of the state vectors of both grids: $\mathbf{x} = \begin{bmatrix} \mathbf{x}_H \\ \mathbf{x}_h \end{bmatrix}$.
Therefore the initial condition $\mathbf{x}^0$, which will be the control variable for the minimization problem, and its background value $\mathbf{x}^b$ (i.e. the first guess at the beginning of the optimization process) follow the same structure.

Cost function

Formally the cost function is the same as in the case of a single grid. It is defined as the sum of the misfit to the first guess and the misfit to the observations on both grids:

$$J(\mathbf{x}^0) = \underbrace{\frac{1}{2}(\mathbf{x}^0 - \mathbf{x}^b)^T \mathbf{B}^{-1}(\mathbf{x}^0 - \mathbf{x}^b)}_{J^b(\mathbf{x}^0)} + \underbrace{\frac{1}{2}\sum_{i=0}^{N}[H_H^i(\mathbf{x}_H^i(\mathbf{x}^0)) - \mathbf{y}_H^i]^T \mathbf{R}_H^{i\,-1}[H_H^i(\mathbf{x}_H^i(\mathbf{x}^0)) - \mathbf{y}_H^i]}_{J_H^{obs}(\mathbf{x}^0)} + \underbrace{\frac{1}{2}\sum_{i=0}^{N}[H_h^i(\mathbf{x}_h^i(\mathbf{x}^0)) - \mathbf{y}_h^i]^T \mathbf{R}_h^{i\,-1}[H_h^i(\mathbf{x}_h^i(\mathbf{x}^0)) - \mathbf{y}_h^i]}_{J_h^{obs}(\mathbf{x}^0)}$$

3. The Multi-grid Adjoint Models

One-way Interaction

Domain Ω_H

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \dfrac{\partial \mathbf{P}}{\partial t} + \left[\dfrac{\partial F}{\partial \mathbf{x}_H}\right]^* .\mathbf{P} + \mathbf{I}_h^H \left[\dfrac{\partial F}{\partial \mathbf{x}_{\partial\omega}}\right]^* .\mathbf{Q} = \mathbf{H}_H^* \mathbf{R}_H^{-1}(\mathbf{H}_H \mathbf{x}_H(t) - \mathbf{y}_H(t)) \\ \mathbf{P}(T) = 0 \end{array}\right. \\ \nabla_{\mathbf{x}_H^0} J^{obs} = -\mathbf{P}(0) \end{array}\right.$$

Domain ω_h

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \dfrac{\partial \mathbf{Q}}{\partial t} + \left[\dfrac{\partial F}{\partial \mathbf{x}_h}\right]^* .\mathbf{Q} = \mathbf{H}_h^* \mathbf{R}_h^{-1}(\mathbf{H}_h \mathbf{x}_h(t) - \mathbf{y}_h(t)) \\ \mathbf{Q}(T) = 0 \end{array}\right. \\ \nabla_{\mathbf{x}_h^0} J^{obs} = -\mathbf{Q}(0) \end{array}\right.$$

Two-way Interaction

Domain Ω_H

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \dfrac{\partial \mathbf{P}}{\partial t} + \left[\dfrac{\partial F}{\partial \mathbf{x}_H}\right]^* .\mathbf{P} + \mathbf{I}_h^H \left[\dfrac{\partial F}{\partial \mathbf{x}_{\partial\omega}}\right]^* .\mathbf{Q} = \mathbf{H}_H^* \mathbf{R}_H^{-1}(\mathbf{H}_H \mathbf{x}_H(t) - \mathbf{y}_H(t)) \\ \mathbf{P}(T) = 0 \end{array}\right. \\ \nabla_{\mathbf{x}_H^0} J^{obs} = -\mathbf{P}(0) \end{array}\right.$$

Domain ω_h

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \dfrac{\partial \mathbf{Q}}{\partial t} + \left[\dfrac{\partial F}{\partial \mathbf{x}_h}\right]^* .\mathbf{Q} + \mathbf{G}_H^h \left[\dfrac{\partial F}{\partial \mathbf{x}_\omega}\right]^* .\mathbf{P} = \mathbf{H}_h^* \mathbf{R}_h^{-1}(\mathbf{H}_h \mathbf{x}_h(t) - \mathbf{y}_h(t)) \\ \mathbf{Q}(T) = 0 \end{array}\right. \\ \nabla_{\mathbf{x}_h^0} J^{obs} = -\mathbf{Q}(0) \end{array}\right.$$

The adjoint formulation adds new interactions between the grids, **in the opposite direction** of the interactions existing in the direct formulation.

4. Multi-grid Background Error Covariance Matrix

The multigrid character of the system can be addressed, by **taking into account the interactions between the grids**. Let assume that background vectors are available on each grid, denoted by $\tilde{\mathbf{x}}_H^b$ and $\tilde{\mathbf{x}}_h^b$. Starting from these vectors, we can define a two-grid background vector $\mathbf{x}^b = (\mathbf{x}_H^b, \mathbf{x}_h^b)^T$, consistent with the interaction operators, as follows:

- $\mathbf{x}_H^b$ is the restriction of $\tilde{\mathbf{x}}_h^b$ in the interior of ω_H and is equal to $\tilde{\mathbf{x}}_H^b$ elsewhere.
- $\mathbf{x}_h^b$ is the interpolation of $\tilde{\mathbf{x}}_H^b$ on the boundary of ω_h and is equal to $\tilde{\mathbf{x}}_h^b$ in the interior.

Given some hypotheses $\mathbf{B}$ can be approximated by $\mathbf{B} = \mathbf{S}\mathbf{S}^T + \mathbf{Q}$, where $\mathbf{S}\mathbf{S}^T$ is a two-grid background error covariance matrix, including both the error covariances on each grid and the covariances between the errors on the different grids due to the interaction operators, and $\mathbf{Q}$ models the covariances corresponding to the errors of the inter-grids operators.

5. Towards a Control of the Inter-grid Interaction Errors

Objectives : limit the negative effects of the propagation between the grids of the error of both models.
Idea : add a new control term, $\epsilon^{ig} = [\epsilon_\omega, \epsilon_{\partial\omega}]^T$, in the inter-grid interaction.

In the general case of two-way interaction, we obtain the new formulation:

Domain Ω_H

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{x}_H}{\partial t} = F_H(\mathbf{x}_H, \mathbf{x}_\omega) \\ \mathbf{x}_H(x,0) = \mathbf{x}_H^0 \\ \mathbf{x}_\omega = G_h^H(\mathbf{x}_h) + \epsilon_\omega \end{array}\right.$$

Domain ω_h

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{x}_h}{\partial t} = F_h(\mathbf{x}_h, \mathbf{x}_{\partial\omega}) \\ \mathbf{x}_h(x,0) = \mathbf{x}_h^0 \\ \mathbf{x}_{\partial\omega} = I_H^h(\mathbf{x}_H) + \epsilon_{\partial\omega} \end{array}\right.$$

In order to regularize the problem, we add to the cost function defined in the previous formulation (§2) a new term $J^{\epsilon^{ig}}(\epsilon^{ig}) = \frac{1}{2}\|\mathbf{K}\epsilon^{ig}\|^2$ which appears as a weak constraint $\mathbf{K}$ on ϵ^{ig}.

The **adjoint models are identical** to the ones defined in the previous formulation.

6. Data Assimilation Experiments

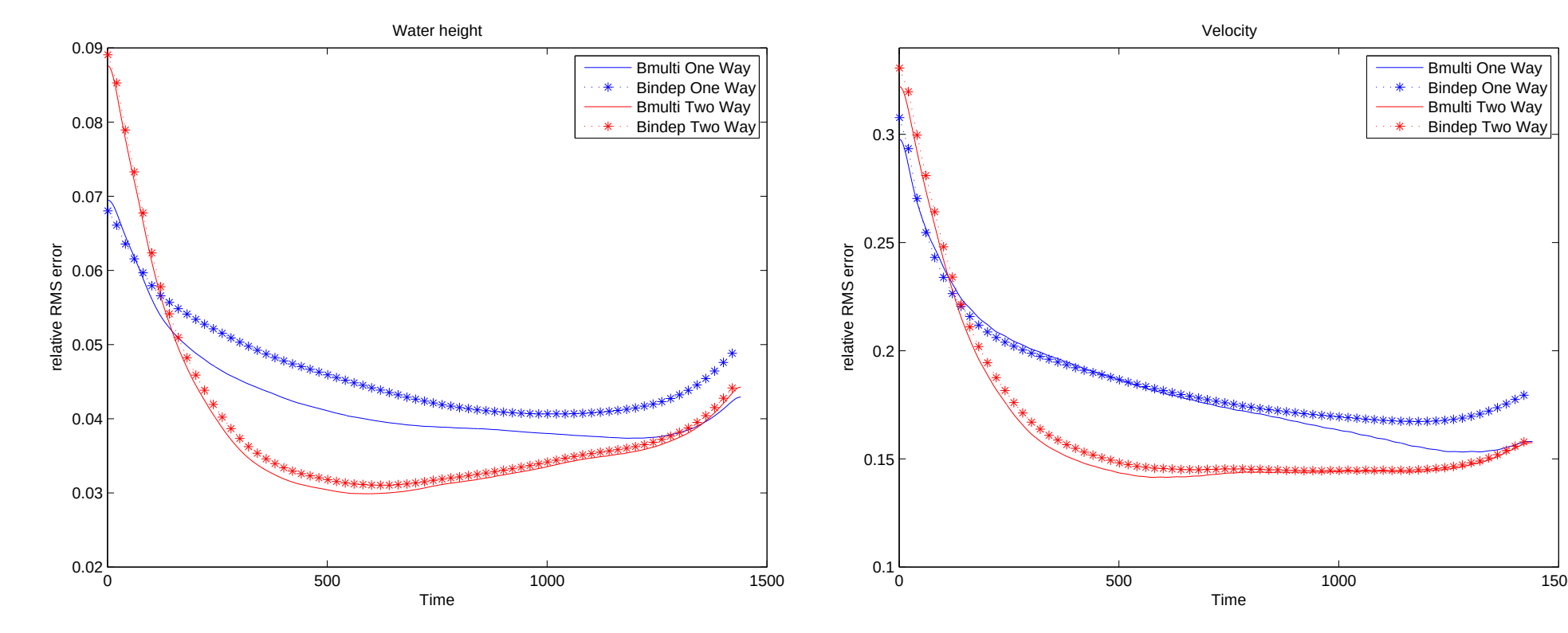
Configuration of the experiments

Double gyre configuration: flat bottom 2D shallow water (ROMS-AGRIF), one local zoom (AGRIF library), spatial refinement factor: 3.
Twin experiment assimilation: one month window, true state: global high resolution solution, observations: water height on the fine grid only, spatial and temporal sampling of the true state.

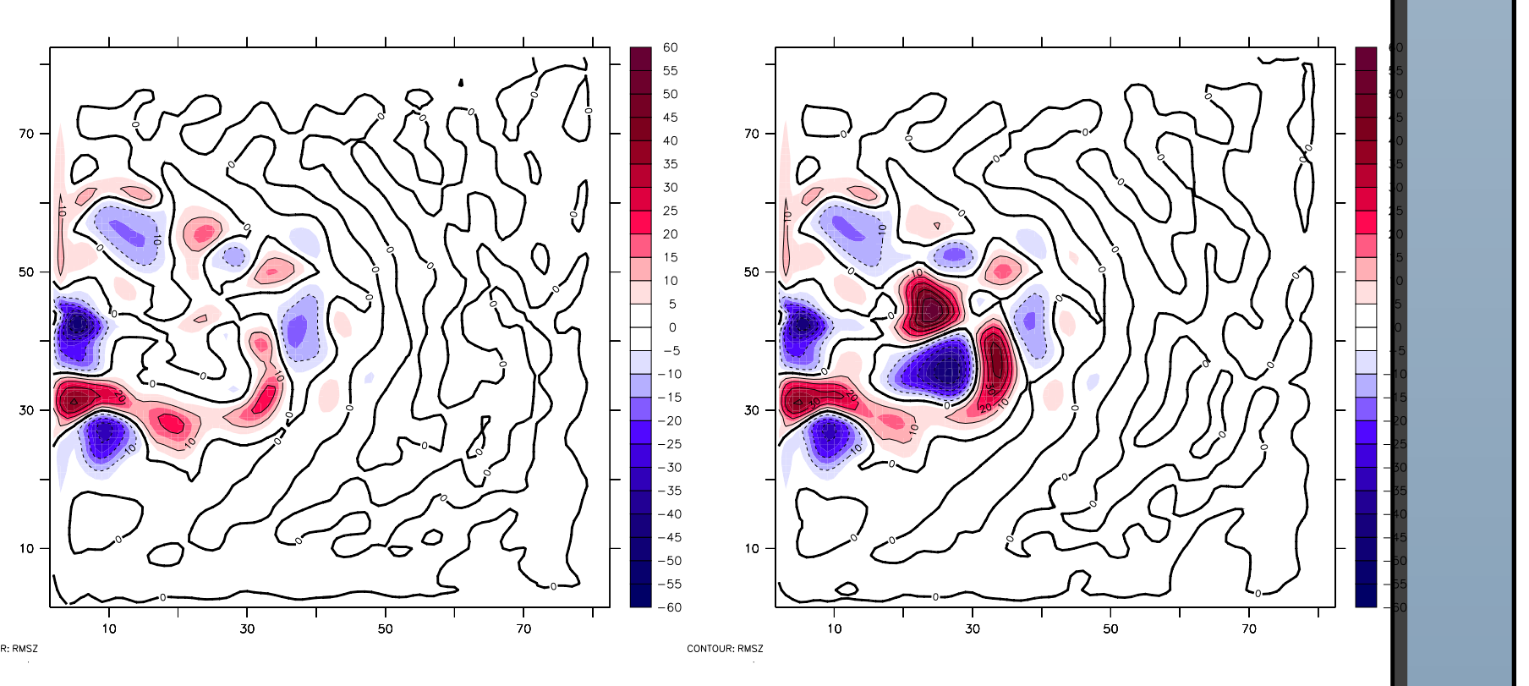
Impact of the background error covariance model

In order to illustrate the impact of the non-diagonal blocks of $\mathbf{B}$, which correspond to the inter-grid covariances, let us consider two versions of $\mathbf{B}$, with ($\mathbf{B}_{multi}$ built as **specified in** §**4**) and without these blocks ($\mathbf{B}_{indep}$ a **block diagonal matrix** obtained by assuming that the first guesses on both grids are independent).

Fine grid ω_h: normalized RMS error of the optimal solutions as a function of the integration time. Water height (left) and velocity (right).

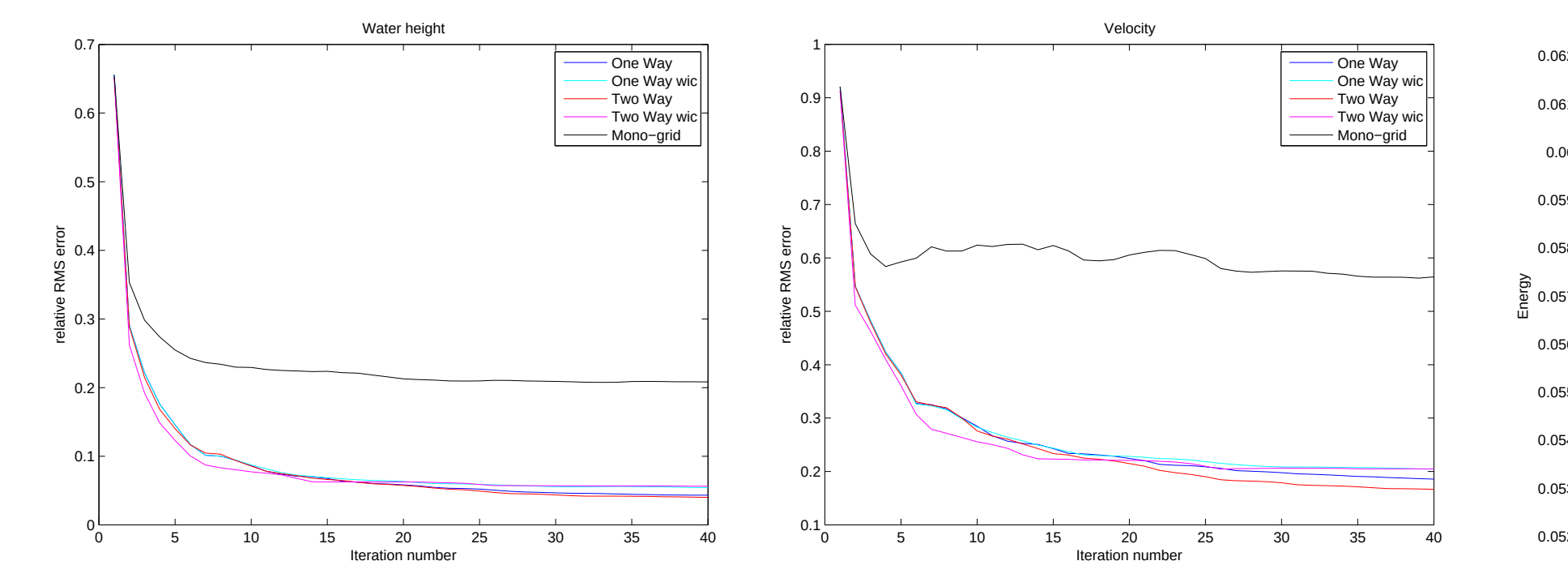


Coarse grid Ω_H: difference between the water height components of the OW optimal solution and the true state at $t = 1000s$. $\mathbf{B}_{multi}$ (left) and $\mathbf{B}_{indep}$ (right).

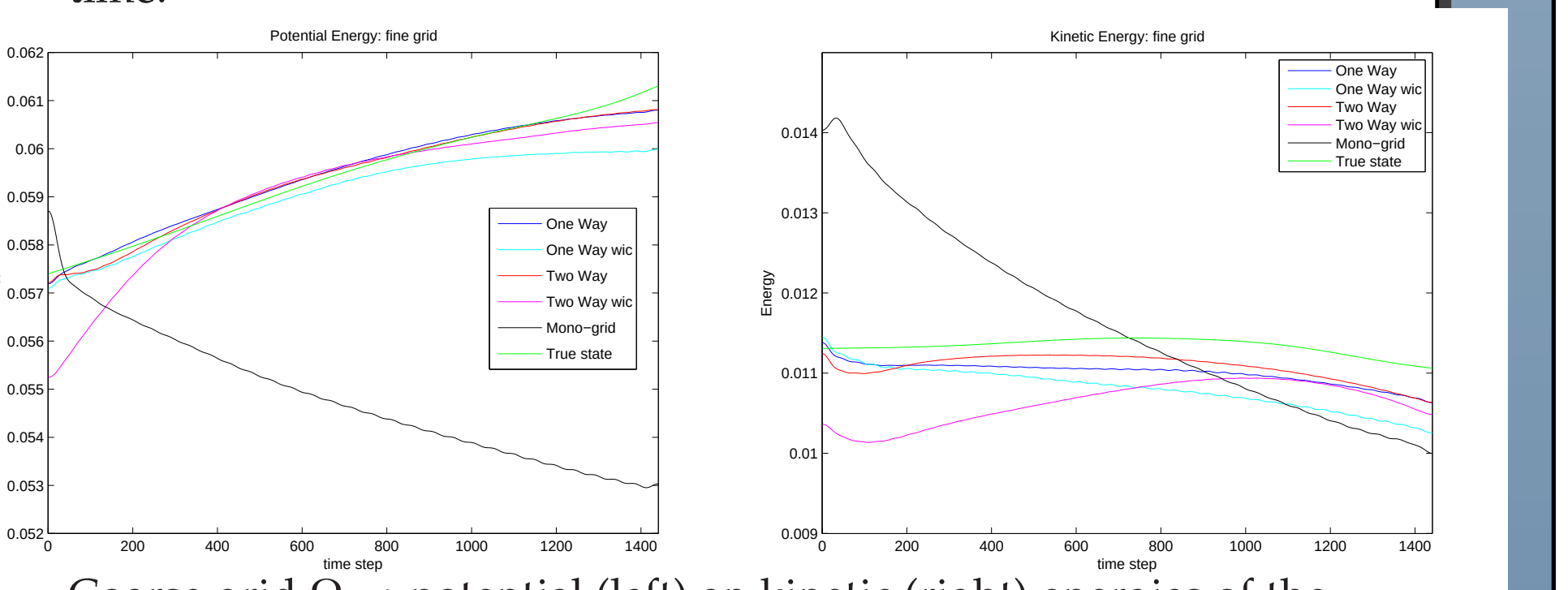


Comparison of the methods

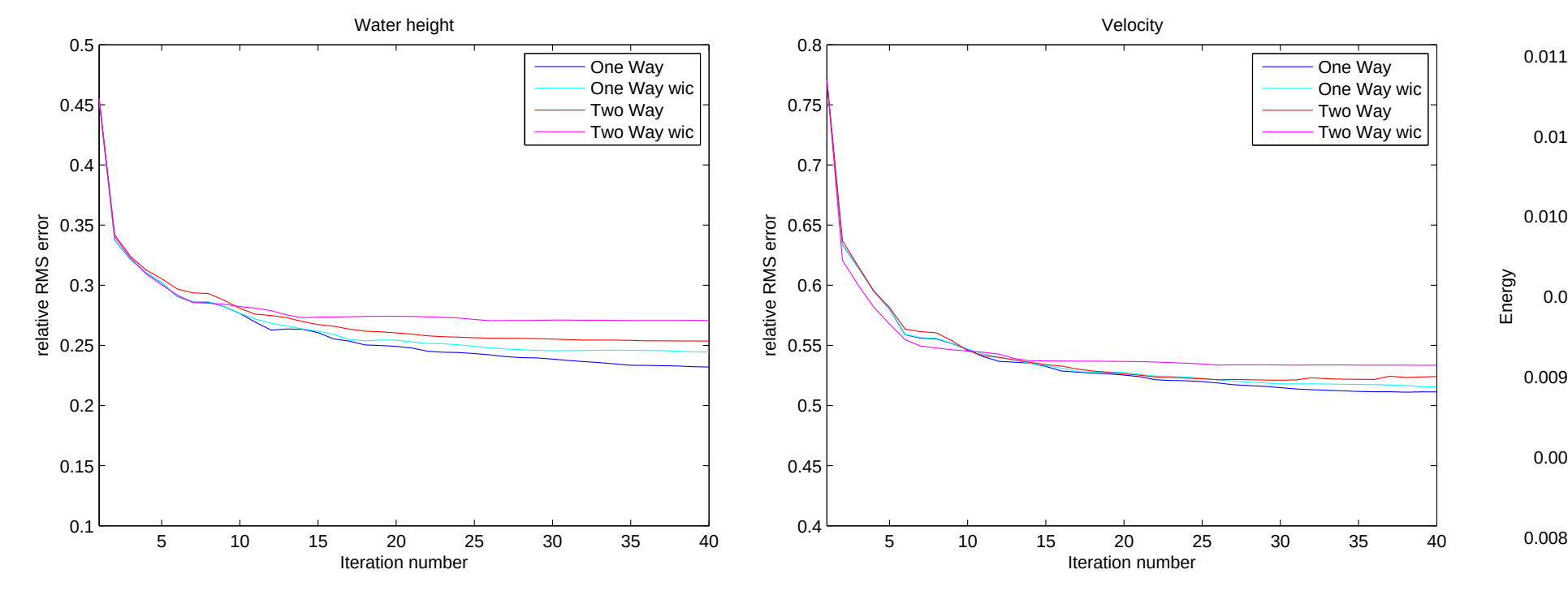
Fine grid ω_h: normalized RMS error as a function of the iteration number. Water height (left) and velocity (right).



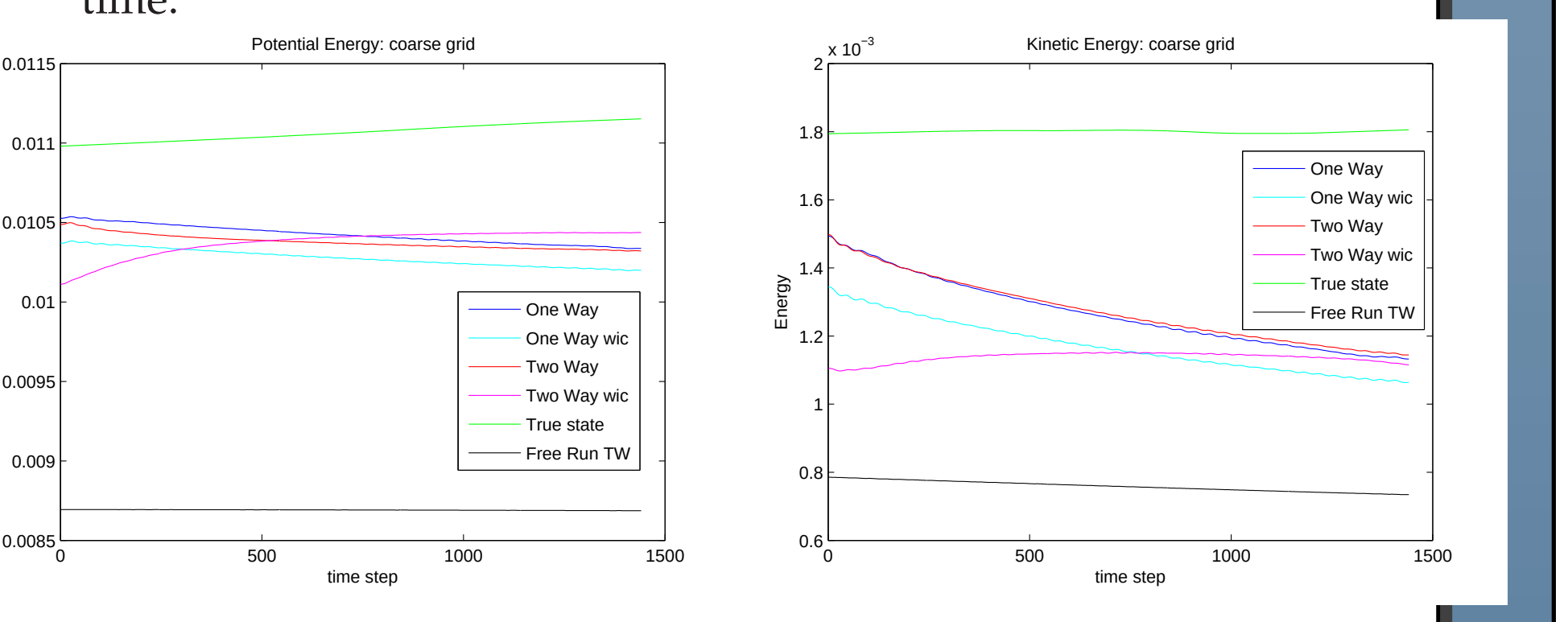
Fine grid ω_h: potential (left) and kinetic (right) energies of the optimal coarse resolution solution as a function of the integration time.



Coarse grid Ω_H: normalized RMS error as a function of the iteration number. Water height (left) and velocity (right).



Coarse grid Ω_H: potential (left) an kinetic (right) energies of the optimal coarse resolution solution as a function of the integration time.



7. Conclusions and open issues

The aim of this study was to analyze in some details the 4D variational data assimilation in the context of locally nested models. The mathematical description of this problem has been performed. We have also dealt with the introduction and the modelling of the multigrid background error covariance matrix $\mathbf{B}$, and the improvement of the nesting method by introducing and controlling what we have called the inter-grid interaction errors. Numerical experiments in a 2D shallow water model with a two-grid configuration have given good results. The results obtained on the high resolution grid are always better than those obtained with a classical 4D-Var algorithm with control of initial and boundary conditions in the local high resolution model with open boundaries. In the case of one-way interaction, our modelling of $\mathbf{B}$ allows a real improvement of the coarse and fine resolution optimal solutions in comparison with solutions obtained with a simple block diagonal matrix. For the case of two-way interaction, this modelling leads only to a slight improvement of the solutions. The *wic* formulation allows a more important decrease of the cost function in comparison to the formulation without this additional control term (not shown). Nevertheless the error of the *wic* optimal solutions are slightly more important at the end of the minimization. We think that it is mainly due to the lack of balance operators on the inter-grid transfer errors, allowing non physical corrections.
Finally the question of the choice of the appropriate density of observation as a function of the grid resolution has not been addressed. This is an important problem to investigate: an observation which is relevant at some given scale can be not relevant any more at an other scale, and can even deteriorate the performance of the data assimilation system.

[1] Debreu L, Simon E, Blayo E. 4D variational data assimilation for locally nested models: optimality system and preliminary numerical experiments. *International Journal for Numerical Methods in Fluids*, 2008; submitted.

[2] Simon E, Debreu L, Blayo E. 4D variational data assimilation for locally nested models: complementary theoretical aspects and application to a 2D shallow water model. *International Journal for Numerical Methods in Fluids*, 2008; submitted.